Real Analysis Ph.D. Qualifying Exam
Temple University
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- Justify your answers thoroughly.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I (Do 3 problems)

I.1. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on R^n. Suppose you have an estimate of the form

$$\int_{R^n} |f_n| \leq c_n \text{ where } c_n \downarrow 0.$$

Can you conclude that $f_n \to 0$ a.e.? If not, what additional condition(s) on $\{c_n\}$ would guarantee this?

I.2. Let $f \in L^\infty[0,1]$ and assume that f is not identically zero. Show that the limit,

$$\lim_{p\to\infty} \frac{\int_0^1 |f|^{p+1} dx}{\int_0^1 |f|^p dx},$$

exists and compute it.

I.3. Let $F(y) = \int_0^\infty e^{-xy} \frac{\sin x}{x} dx$, $y > 0$.

(a) Show that F is continuous on $(0, \infty)$.

(b) Prove $F'(y) = -\int_0^\infty e^{-xy} \sin x dx$, $y > 0$.

I.4. Let $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denote the symmetric difference of sets A and B. Let A_n and B_n be measurable subsets of R. Suppose $\lambda(A_n \triangle B_n) = 0$, for all n, where λ is the Lebesgue measure.

(a) Show that $\lambda[(\bigcup_{n=1}^\infty A_n) \triangle (\bigcup_{n=1}^\infty B_n)] = 0$.

(b) Show that

$$\lambda[(\limsup_{n\to\infty} A_n) \triangle (\limsup_{n\to\infty} B_n)] = 0,$$

where $\limsup_{n\to\infty} A_n = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_n$.

Part II (Do 2 problems)

II.1. Let S be a measurable subset of R^2. Assume for every $x \in S$ there exists a sequence of cubes $\{Q_k(x)\}$ centered at x with side lengths tending to zero such that

$$|S \cap Q_k(x)| \leq \frac{1}{2} |Q_k(x)|.$$

Show that $|S| = 0$.

II.2. A sequence of functions $\{f_n\} \in L^1[0,1]$ is said to be *uniformly integrable* if

$$\lim_{c\to\infty} \sup_{n\geq 1} \int_{x\in[0,1]:|f_n(x)|>c} |f_n(x)|dx = 0.$$

If for such a sequence it holds that $f_n \to f$ almost everywhere for some measurable f, prove that $f_n \to f$ in $L^1[0,1]$ norm.

II.3. Prove that

$$\int_0^\infty \frac{\sin t}{e^t - x} dt = \sum_{n=1}^\infty \frac{x^{n-1}}{n^2+1},$$

for $-1 < x < 1$.