Real Analysis Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Let x_k be a sequence in a metric space (X, d) such that $\sum_{k=1}^{\infty} d(x_k, x_{k+1}) < \infty$. Prove that x_k is a Cauchy sequence.

2. Prove that the function
$$F(x) = \int_0^{+\infty} \frac{\cos(x\,t^2)}{1+t^2}\,dt$$
is continuous for all $x \in \mathbb{R}$.

3. Let $f_n(x) = n\,x\,e^{-n\,x^2}$ on $[0, +\infty)$. Prove that
   (a) f_n converges to zero pointwise in $[0, +\infty)$
   (b) f_n does not converge uniformly in $[0, +\infty)$
   (c) f_n converges in measure on $[0, +\infty)$
   (d) $\int_0^{\infty} f_n(x)\,dx = \frac{1}{2}$.

   HINT for (c): may use that $e^z \geq z^2/2$ for all $z \geq 0$.

4. Let $f \in C^1[0, +\infty)$ such that $f(x) \to 0$ as $x \to +\infty$. Prove that
$$\int_0^{\infty} f(x)^2\,dx \leq 2 \left(\int_0^{\infty} x^2 f(x)^2\,dx \right)^{1/2} \left(\int_0^{\infty} f'(x)^2\,dx \right)^{1/2}.$$
   HINT: write $f(x)^2 = -\int_x^{\infty} \left(f(t)^2\right)'\,dt$.

Part II. (Do 2 problems)

1. Let $p \geq 1$, $f_k \in L^p(\mathbb{R}^n)$ with $f_k \to f$ a.e., and $g(x) := \sup_k |f_k(x)| \in L^p(\mathbb{R}^n)$. Prove that $f \in L^p(\mathbb{R}^n)$ and $f_k \to f$ in $L^p(\mathbb{R}^n)$.

2. Let f_k be absolutely continuous functions in $[a, b]$ such that $f_k(a) = 0$ for all k. Suppose that f_k' is a Cauchy sequence in $L^1[a, b]$. Prove that there exists f absolutely continuous on $[a, b]$ such that $f_k \to f$ uniformly in $[a, b]$.

3. Let χ_E denote the characteristic function of a bounded set $E \subset \mathbb{R}^n$. Prove that
   (a) the function $\phi(y) = \int_{\mathbb{R}^n} \chi_E(x)\,\chi_E(y+x)\,dx$ is continuous for each $y \in \mathbb{R}^n$.
   (b) If $|E| > 0$, then show that (a) implies that 0 is an interior point of the set $E - E = \{x - y : x, y \in E\}$.