Real Analysis Ph.D. Qualifying Exam
Temple University
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Part I. (Select 3 questions.)

1. Let $f : [a, b] \to \mathbf{R}$ be a bounded function and set

$$M = \sup_{[a,b]} f(x), \quad m = \inf_{[a,b]} f(x), \quad M^* = \sup_{[a,b]} |f(x)|, \quad m^* = \inf_{[a,b]} |f(x)|.$$

   Prove that $M^* - m^* \leq M - m$.

2. Let $f_n(x) = \dfrac{n^{3/2}x}{1 + n^2x^2}$ for $0 \leq x < +\infty$. Prove that

   (a) $f_n(x) \to 0$ as $n \to \infty$ for each $x \in [0, +\infty)$.

   (b) f_n does not converge uniformly in $[0, +\infty)$.

   (c) $f_n \to 0$ in measure as $n \to \infty$ in $[0, +\infty)$.

3. Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|}$$

   converges for each $x \in \mathbf{R}$ and the sum defines a Lipschitz function.

4. For a given set $E \subset \mathbf{R}^n$ consider the open set $O_k = \{x : \text{dist}\,(x, E) < 1/k\}$. Prove that $|E| = \lim_{k\to\infty} |O_k|$ for E compact. Prove that this may be false for E closed and unbounded.

Part II. (Select 2 questions.)

1. Let $1 \leq p < \infty$ and $f_n, f \in L^p(E)$ with $\|f_n - f\|_p \to 0$ as $n \to \infty$. Prove that

   (a) $\forall \epsilon > 0\ \exists n_0$ and $A \subset E$ measurable with $|A| < \infty$ such that $\displaystyle\int_{E\setminus A} |f_n(x)|^p \, dx \leq \epsilon$ for all $n \geq n_0$; and

   (b) $\forall \epsilon > 0\ \exists n_0, \delta > 0$ such that if $|F| < \delta$, then $\displaystyle\int_F |f_n(x)|^p \, dx \leq \epsilon$ for all $n \geq n_0$.

2. If $|f_k| \leq g$ a.e. with g integrable in E, and $f_k \to f$ in measure in E, then prove that

$$\int_E f(x)\, dx = \lim_{k\to\infty} \int_E f_k(x)\, dx.$$

3. Prove that if E_k is a sequence of measurable sets in $\mathbf{R}^n$ with $\sum_{k=1}^{\infty} |E_k| < \infty$, then $|\limsup E_k| = 0$.