Mathematics Real Analysis Ph.D. Qualifying Exam — January 16, 2009

PART I (select three questions)

1. Let $f : \mathbb{R} \to \mathbb{R}$ be $C^n(\mathbb{R})$ for some $n \geq 0$. Prove that if $f^{(k)}(0) = 0$, for all $0 \leq k \leq n$, then $\dfrac{f^{(k)}(x)}{|x|^{n-k}} \to 0$ as $x \to 0$, for all $0 \leq k \leq n$.

2. Prove that on $C[0,1]$ the norms $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$ and $\|f\|_1 = \int_0^1 |f(x)|\, dx$ are not equivalent.

3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions. Suppose that f_n' converges uniformly to a function g in $\mathbb{R}$, and $f_n(0)$ converges as $n \to \infty$. Prove that $f_n(x)$ converges for each $x \in \mathbb{R}$.

4. Let $f_n : E \to \mathbb{R}$ be a sequence of measurable functions. Prove that the set

$$\{x \in E : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

is measurable.

PART II (select two questions)

1. Let $f \in L^1(0, +\infty)$ be nonnegative. Prove that

$$\frac{1}{n}\int_0^n x\, f(x)\, dx \to 0, \qquad \text{as } n \to \infty.$$

Hint: Write $\dfrac{1}{n}\displaystyle\int_0^n x\, f(x)\, dx = \int_0^a \frac{x}{n} f(x)\, dx + \int_a^n \frac{x}{n} f(x)\, dx$, and pick a sufficiently large.

2. Let $f_n(x) = n \sin\left(\dfrac{x}{n}\right)$. Prove that:

   (a) f_n converges uniformly on any finite interval. Hint: $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$ for all x.

   (b) f_n does not converge uniformly on $\mathbb{R}$.

   (c) f_n does not converge in measure on $\mathbb{R}$. Hint: the interval $(n\pi, (n+1)\pi)$ is contained in the set $|f_n(x) - x| > \epsilon$.

3. Let r_n be the sequence of all rational numbers and

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{|x - r_n|^{1/2}}.$$

Prove that

   1. $\displaystyle\int_a^b f(x)\, dx < \infty$,
   2. $\displaystyle\int_a^b f(x)^2\, dx = +\infty$,

for all $a < b$.