Real Analysis Qualifying Exam
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All integrals are Lebesgue integrals; do not use the Riemann integral or use any facts relating to it. Justify your reasoning carefully and clearly.

Part I

Please select 3 of these problems.

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at all x and differentiable at all nonzero x. Assume $f'(x) \to 0$ as $x \to 0$. Show that f is differentiable at 0.

2. Suppose $f(x)$ is defined on $[-1, 1]$ and $f'''(x)$ is continuous. Show that

   1. There is a continuous function $g : [-1, 1] \to \mathbb{R}$ such that
   $$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + g(x)x^3.$$
   2. Show that the series
   $$\sum_{n=1}^{\infty} (n\,[f(1/n) - f(-1/n)] - 2f'(0))$$
   converges.

3. Let $A \subset \mathbb{R}$ be compact, let $x_0 \in A$, and let $(x_n) \subset A$ be a sequence. Assume every convergent subsequence of (x_n) converges to x_0.

   1. Show that (x_n) converges.
   2. Show that if A is not compact, the result in part 1 is not necessarily true.

4. Given a sequence (a_n), let $a_n^* = \sup\{a_k : k \geq n\}$, $n \geq 1$, be the corresponding upper sequence. Show that **either** the sequence (a_n^*) is eventually constant **or** $a_n^* = \max\{a_k : k \geq n\}$ for all $n \geq 1$.

Part II

Please select 2 of these problems.

1. Let $A = \{(a, b) : a > 0, b > 0\}$. Show that
$$M(a, b) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}\,d\theta$$
is a continuous function on A.

2. Suppose f and g are continuous functions on $\mathbb{R}$ and $g(x + 1) = g(x)$. Show that
$$\lim_{n\to\infty}\int_0^1 f(x)g(nx)\,dx = \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right).$$
(Write $[0, 1]$ as a union of n sub-intervals.)

3. Let $f : [0, 1] \to \mathbb{R}$ be continuously differentiable with $f(0) = 0$. Show that
$$\sup_{0\leq x\leq 1} |f(x)| \leq \left(\int_0^1 f'(x)^2\,dx\right)^{1/2}.$$