PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

January 1996

Part I. Do three (3) of these problems.

I.1. Let $\{a_n\}$ be a sequence of real numbers with the following property: there is a constant $0 < K < 1$ such that

$$|a_{n+2} - a_{n+1}| \leq K|a_{n+1} - a_n| \quad \text{for all } n \geq N_0.$$

Prove that $\{a_n\}$ converges.

I.2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $x_1, \dots, x_n \in [a, b]$. Show that there exists $z \in [a, b]$ such that

$$f(z) = \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

I.3. Give an example of a function $f \in L^p(\mathbb{R})$, $p \geq 1$, such that

$$\lim_{x \to \infty} f(x) \neq 0.$$

I.4. (1) Let $\{f_n\}$ be a subsequence of $L^1(\mathbb{R})$ such that $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$. Show that $\sum_{n=1}^{\infty} f_n$ converges absolutely a.e.

(2) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{A_n\}$ be a subsequence of $\mathcal{A}$. Show that if $\sum_{n=1}^{\infty} \mu(A_m) < \infty$ then $\mu(\limsup A_n) = 0$, where $\limsup A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$.

Part II. Do two (2) of these problems.

II.1. Let

$$F(y) = \int_0^{\infty} e^{-2x} \cos(2xy)dx, \quad y \in \mathbb{R}.$$

Show that F satisfies the differential equation

$$F'(y) + 2yF(y) = 0.$$

Justify the differentiation under the integral sign.

II.2. Let f be a real valued function defined on a closed bounded interval $[a, b]$. Establish the following:

(1) If f is continuous, f need not be of bounded variation. Consider

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

(2) If f satisfies a Lipschitz condition, that is, $|f(x) - f(y)| \leq M|x - y|$ for some positive number M and all x, $y \in [a, b]$, then f is absolutely continuous.
(3) If f' exists everywhere and is bounded on $[a, b]$, then f is absolutely continuous.

II.3. Let H be a Hilbert space and $y_0 \in H$. Show that there exists $\Lambda \in H^*$ (bounded linear functional on H) different from zero such that

$$\Lambda(y_0) = \|\Lambda\|_{H^*} \|y_0\|.$$

(Hint: Either apply the Hahn-Banach theorem with the sublinear functional $p(x) = \|x\|$, or construct a bounded linear functional in terms of y_0).