PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

August 2001

Part I. Do three (3) of these problems.

I.1. Let $F \subset \mathbb{R}^n$ be a closed set, and $r > 0$. Let

$$G = \{y \in \mathbb{R}^n : |x - y| = r \text{ for some } x \in F \text{ depending on } y\}.$$

Prove that G is closed.

I.2. Let $f : \mathbb{R} \to \mathbb{R}$ be convex. Suppose that

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \frac{f(x)}{x} = 0.$$

Prove that f is constant.
Hint: use (do not prove) the following fact from convexity: for each $y \in \mathbb{R}$ there exists $p \in \mathbb{R}$ such that $f(x) \geq p\,(x - y) + f(y)$ for all $x \in \mathbb{R}$.

I.3. Let $f \in C^1[0, \infty)$ with $f(x) \to 0$ as $x \to \infty$. Prove that

$$\int_0^\infty f(x)^2\,dx \leq 2 \left(\int_0^\infty x^2 f(x)^2\,dx \right)^{1/2} \left(\int_0^\infty f'(x)^2\,dx \right)^{1/2}.$$

Hint: write $f(x)^2 = -\int_x^\infty \left(f(t)^2\right)'\,dt$.

I.4. Let $f : [0, \infty) \to \mathbb{R}$ be $C^1[0, \infty)$ with $f(0) = 0$. Suppose there exists $m > 0$ such that

$$0 \leq f'(x) \leq m\,f(x), \qquad \text{for all } 0 \leq x < \infty.$$

Prove that $f(x) = 0$ for $0 \leq x < \infty$.

Part II. Do two (2) of these problems.

II.1. Let $p \geq 1$. Suppose $f_k \in L^p(\mathbb{R}^n)$, $\sup_k |f_k| \in L^p(\mathbb{R}^n)$, and $f_k \to f$ a.e. Prove that $f \in L^p(\mathbb{R}^n)$ and $f_k \to f$ in $L^p(\mathbb{R}^n)$.

II.2. Let

$$f_n(x) = \frac{1}{\left|x - \frac{1}{n}\right|^{1/2}}$$

on the interval $(0, 1)$. Prove that

1. (1) f_n converges pointwise on $(0, 1)$;
2. (2) f_n converges in measure on $(0, 1)$;
3. (3) f_n converges in $L^1(0, 1)$;
4. (4) there does not exist $g \in L^1(0, 1)$ such that $f_n(x) \leq g(x)$ for a.e. $x \in (0, 1)$ and for all n.

Hint for (4): calculate $\int_{1/(n+1)}^{1/n} h(x)\,dx$, where $h(x) = \max\{f_n(x), f_{n+1}(x)\}$.

II.3. Let $f, g \in L^2[0, 1]$ be extended as periodic functions to $\mathbb{R}$, i.e., $f(x + 1) = f(x)$ and $g(x + 1) = g(x)$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)\, g(nx)\, dx = \int_0^1 f(x)\, dx \int_0^1 g(x)\, dx.$$

Hint: consider first the case when g is a trigonometric polynomial, $g(x) = \sum_{k=0}^{N} a_k\, e^{2\pi i k x}$, and use the Riemann-Lebesgue theorem asserting that $\int_0^1 f(x)\, e^{2\pi i m x} dx \to 0$ as $m \to \infty$. For the general case of $g \in L^2[0, 1]$ approximate g by a trigonometric polynomial in L^2-norm.