Ph.D. Qualifying Examination in Partial Differential Equations
Temple University
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Part I. (Do 3 problems in this part)

1. Let u be a non constant harmonic function on a domain Ω. Prove that if $B \subset \Omega$ is a ball, then $u(B)$ is an open set in the real line.

2. Let Ω be a bounded, smooth domain in $\mathbb{R}^n$ and $u \in C^2(\bar{\Omega})$ a solution of

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = C \qquad \text{in } \Omega$$

with $\nu \cdot \left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \alpha$ on $\partial\Omega$, where ν is the outer unit normal and $|\alpha| < 1$ is a constant. Calculate the value of the constant C.

3. Let $B = \{x \in \mathbb{R}^3 : |x| < 1\}$. Show that $u(x) = |x|^{-1}$ belongs to the Sobolev space $W^{1,1}(B)$. (Note: For full credit, you should compute the weak derivative of u with justification).

4. Suppose $a : \mathbb{R} \to \mathbb{R}$ is a C^1 function. Prove that there is a solution u of

$$a(u)\, u_x + u_y = 0, \qquad u(x, 0) = x.$$

Part II. (Do 2 problems in this part)

1. Let $u \in C^2(\bar{\Omega})$ satisfying $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Prove the maximum principle

$$\max_{\Omega} |u| \le \max_{\partial\Omega} |g| + C \max_{\Omega} |f|,$$

where C is a constant depending only on Ω and the dimension.

HINT: let $M = \max_{\Omega} |f|$ and consider the function $v(x) = \frac{M}{2n}|x|^2 - u(x)$; v is subharmonic and apply the max principle to v.

2. Let $f(x)$ be a bounded function on $\mathbb{R}$ which is also in $L^1(\mathbb{R})$. Write a formula for a solution of

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x, 0) = f(x), & x \in \mathbb{R} \end{cases}$$

that satisfies (a) $\|u\|_{L^\infty} \le C\|f\|_{L^\infty}$ and (b) $\sup_{t>0} \|u(.,t)\|_{L^1} \le \|f\|_{L^1}$ where C is some constant. Show that u does satisfy the estimates (a) and (b).

3. Consider the $n-$ dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - c^2\Delta u + \alpha u_t = 0 & \text{on } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x), & u_t(x, 0) = h(x) \end{cases}$$

where g and h are C^2 of compact support and $\alpha \geq 0$. Use the function

$$E(t) = \int_{\mathbb{R}^n} (u_t(x,t)^2 + c^2|\nabla u(x,t)|^2)\, dx$$

to prove that any C^2 solution u is determined by its Cauchy data g and h. You may assume that for each $t > 0$, the function $x \to u(x,t)$ has a compact support. (Here ∇u denotes the $x-$gradient).