Sample problems for Geometry–Topology qualifying exam

Summer 2013

The following list of problems is much longer than a typical qualifying exam would be. This is deliberate: the goal is to have a significant sample of study problems.

Smooth manifolds

1. Let $SL(n, \mathbb{R})$ be the special linear group, i.e. the set of all $n \times n$ matrices with unit determinant. Give a rigorous proof that $SL(n, \mathbb{R})$ is a smooth manifold. What is its dimension?

2. Let M be a smooth manifold. Give a careful definition of the tangent bundle TM, and prove that the manifold TM is orientable.

3. If M and N are smooth manifolds, where N is non-orientable, prove that $M \times N$ is non-orientable.

4. Let M be a smooth manifold, and $f : M \to \mathbb{R}$ be a continuous function, where $f(x) > 0$ for all x. Prove that there exists a smooth function $g : M \to \mathbb{R}$, such that $0 < g(x) < f(x)$ for all x.

5. Prove the following special case of Sard's theorem: if $f : M \to N$ is a smooth map, where $\dim(M) < \dim(N)$, then $f(M)$ has measure 0 in N.

6. Let ω and η be smooth 3–forms on S^7. Prove that

$$\int_{S^7} \omega \wedge d\eta = \int_{S^7} d\omega \wedge \eta.$$

7. Let M be a smooth manifold of dimension $2n$, and let ω be a smooth 2–form such that $\omega^{\wedge n}$ is nowhere vanishing. Let $\{x^i\}_{1 \leq i \leq 2n}$ be local coordinates on a subset $U \subset M$ and let $||\alpha^{ij}(x)||$ be the inverse of the matrix $||\omega_{ij}(x)||$ which represents ω on the coordinate subset U. Prove that the matrix-valued functions $||\alpha^{ij}(x)||$ define on M a contravariant antisymmetric tensor of rank 2. Prove that the two-form ω is closed if and only if the following operation

$$\{\ ,\ \} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$$

$$\{a, b\} := \sum_{1 \leq i,j \leq 2n} \alpha^{ij} (\partial_{x^i} a)(\partial_{x^j} b)$$

satisfies the Jacobi identity:

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0\,, \qquad \forall\ \ a, b, c \in C^\infty(M)\,.$$

Algebraic topology

1. Consider the annulus $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$. Construct the quotient space

$$X = A/\sim, \quad \text{where} \quad e^{i\theta} \sim e^{i(\theta+2\pi/3)} \quad \text{and} \quad 2e^{i\theta} \sim 2e^{i(\theta+2\pi/4)}.$$

In words: on the inner circle of A, we identify points that are 120° apart, while on the outer circle we identify points that are 90° apart.

Compute $\pi_1(X)$. *Hint:* cut A along the circle of radius 1.5.

2. Let $f : \mathbb{RP}^2 \to S^1$ be a continuous map. Prove that f is homotopic to the constant map.

3. Let $A = S^1 \times [0, 1]$ be an annulus, and let $B = S^1 \times S^2$. Join the two spaces together by a homeomorphism from ∂A to the union of two circles $(S^1 \times \{x_0\}) \cup (S^1 \times \{x_1\})$, oriented in the same direction. Compute the homology groups of the resulting topological space Y.

4. Let V be a solid torus, and T its boundary torus. Compute the relative homology groups $H_i(V, T)$.

5. State and prove the Brouwer fixed point theorem in n dimensions. You may assume standard facts about the homology groups of standard spaces.

6. Let M be a smooth manifold. Let $C_k = C_k^\infty(M, \mathbb{R})$ be the set of k–dimensional smooth singular chains, and let $\Omega^k = \Omega^k(M)$ be the set of k–dimensional differential forms on M. Define a homomorphism $h : \Omega^k \to C^k = \mathrm{Hom}(C_k, \mathbb{R})$ via

$$h(\omega)(\sigma) = \int_\sigma \omega.$$

Prove that $h : \Omega^k \to C^k$ is a homomorphism of cochain complexes, and therefore gives a homomorphism of cohomology groups.

7. Let Conf_n be the following submanifold of $\mathbb{C}^n$:

$$\mathrm{Conf}_n := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \quad \text{such that} \quad a \neq b \Rightarrow z_a \neq z_b\}. \tag{1}$$

Prove that for every pair (a, b) of distinct indices the 1-form

$$\eta_{ab} := \frac{d(z_a - z_b)}{z_a - z_b}$$

represents a non-zero cohomology class in $H^1(\mathrm{Conf}_n, \mathbb{C})$. Prove that for every triple (a, b, c) of distinct indices the 2-form

$$\eta_{ab}\eta_{bc} + \eta_{bc}\eta_{ca} + \eta_{ca}\eta_{ab}$$

is exact. *Hint: Given a triple (a, b, c) of distinct indices, prove that* Conf_n *is homotopy equivalent to its submanifold defined by equations $z_a = 0$ and $|z_b - z_a| = 1$.*

8. It is known that every finitely presented group is the fundamental group of some smooth compact 4–manifold. Let M be a closed (compact without boundary) orientable smooth 4–manifold with fundamental group Γ, and suppose that Γ is generated by elements of finite order. Prove that every smooth vector field on M has a zero.