Comprehensive Examination in Geometry & Topology
Department of Mathematics, Temple University

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Part I. Solve three of the following problems.

I.1 Prove that any continuous map from the real projective plane $\mathbb{RP}^2$ to the n-dimensional torus $T^n = (S^1)^n$ is nullhomotopic ($n \geq 1$).

I.2 Let n be an integer ≥ 2 and $\mathrm{SL}_n(\mathbb{R})$ be the following subset in the set $\mathrm{Mat}_{n\times n}(\mathbb{R})$ of $n \times n$ matrices with real entries:

$$\mathrm{SL}_n(\mathbb{R}) := \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid \det(A) = 1\}.$$

(a) Prove that $\mathrm{SL}_n(\mathbb{R})$ is a submanifold of $\mathrm{Mat}_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $n^2 - 1$.

(b) Prove that the assignment $A \mapsto A^{-1}$ defines a diffeomorphism from $\mathrm{SL}_n(\mathbb{R})$ onto itself.

I.3 Let X_n be the rose with n petals, i.e., a graph with one vertex and n edges.

1. Compute $\pi_1(X_n)$ (e.g., with base point the vertex).
2. Draw all the equivalence classes of 2-sheeted covers of X_2 relative to your base point (*hint: there are* 3 *of them*).

I.4 Let $\mathrm{Conf}_2(\mathbb{R}^2)$ be the configuration space of pairs of points in $\mathbb{R}^2$:

$$\mathrm{Conf}_2(\mathbb{R}^2) := \big\{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \mathbf{x}_1 \neq \mathbf{x}_2\big\}$$

with the obvious manifold structure inherited from $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. Consider the map $f : \mathrm{Conf}_2(\mathbb{R}^2) \to S^1$ defined by the formula

$$f(\mathbf{x}_1, \mathbf{x}_2) := \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|}$$

and let ω be a 1-form on S^1 satisfying

$$\int_{S^1} \omega \neq 0.$$

Prove that the pull back $f^*(\omega)$ is a closed 1-form on $\mathrm{Conf}_2(\mathbb{R}^2)$ that is not exact.

Part II. Solve two of the following problems.

II.1 A *knot* K in the 3-sphere S^3 is the image a smooth embedding $f : S^1 \to S^3$. Set $M = S^3 \smallsetminus N(K)$, where $N(K)$ is a small tubular neighborhood of K in S^3.

(a) Calculate the integer homology groups $H_*(M; \mathbb{Z})$ using the Mayer–Vietoris sequence. You may assume you know the homology groups of tori and spheres.

(b) Conclude from (a) that $\pi_1(M)$ is infinite.

II.2 Let A and B be two distinct two dimensional tori and f be a degree d map

$$f : \; S^1 \times \{x_0\} \subset A \;\to\; S^1 \times \{y_0\} \subset B.$$

Compute the fundamental group of the space

$$X := B \sqcup_f A$$

obtained by attaching A to B along $S^1 \times \{x_0\} \subset A$ via f.

II.3 Is there a smooth vector field on the 2-sphere S^2 that vanishes at exactly one point? Either give an example, or prove that no such vector field exists.