Ph.D. Comprehensive Examination in Complex Analysis
Department of Mathematics, Temple University

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Part I. Do three of these problems.

I.1 (i) Show that $u(x, y) = \log(x^2 + y^2)$ is harmonic in $\mathbb{C} \setminus \{0\}$.

(ii) Find a harmonic conjugate of $u(x, y)$ in $\mathbb{C} \setminus \{x \in \mathbb{R} : \ x \geq 0\}$. Please justify your answer briefly

(iii) Show that $u(x, y)$ has no harmonic conjugate in $\mathbb{C} \setminus \{0\}$.

I.2 Let $f(z)$ be an entire function.

(i) Suppose that for all z sufficiently large $|f(z)| \leq \dfrac{|z|^4}{1 + |z|^2}$. Show that $f(z)$ is a polynomial of degree less or equal to 2.

(ii) Suppose $|f(z)| \leq \dfrac{|z|^4}{1 + |z|^2}$ for all $z \in \mathbb{C}$. Show that $f(z) = 0$.

I.3 Let $f(z)$ and $g(z)$ be analytic in $B(z_0, r) \setminus \{z_0\}$ for some $r > 0$.

(i) Suppose $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z) = 0$ and that $\lim_{z \to z_0} \dfrac{f'(z)}{g'(z)}$ exists. Show that $\lim_{z \to z_0} \dfrac{f(z)}{g(z)}$ exists and equals $\lim_{z \to z_0} \dfrac{f'(z)}{g'(z)}$.

(ii) Suppose $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z) = \infty$ and that $\lim_{z \to z_0} \dfrac{f'(z)}{g'(z)}$ exists. Show that $\lim_{z \to z_0} \dfrac{f(z)}{g(z)}$ exists and equals $\lim_{z \to z_0} \dfrac{f'(z)}{g'(z)}$.

I.4 Use the calculus of residues to find $\displaystyle\int_0^\infty \frac{\log x \, dx}{(x^2 + 1)^2}$.

Part II. Do two of these problems.

II.1 Let γ be a rectifiable closed path and $f(z)$ a meromorphic function in $\mathbb{C}$ such that no zeros or poles of $f(z)$ lie on γ.

(i) Show that there exist at most finitely many zeros $z_1, \ldots, z_k$ and at most finitely many poles $p_1, \ldots, p_\ell$ of $f(z)$ such that $n(\gamma, z_i) \neq 0$ and $n(\gamma, p_j) \neq 0$.

(ii) Show that $\displaystyle\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{i=1}^{k} n(\gamma, z_i) m_i - \sum_{j=1}^{\ell} n(\gamma, p_j) n_j$ where m_i is the order of the zero of $f(z)$ at z_i and n_j is the order of the pole of $f(z)$ at p_j.

II.2 Let $\{f_n(z)\}$ be a sequence of functions analytic in the open unit disc $B(0,1)$ and continuous on the closed unit disc $\bar{B}(0,1)$.

(i) Suppose the sequence $\{f_n(z)\}$ converges uniformly on $\partial B(0,1)$. Show that $\{f_n(z)\}$ converges uniformly in $\bar{B}(0,1)$.

(ii) Let $f(z) = \lim_{n\to\infty} f_n(z)$. Show that $f(z)$ is analytic in $B(0,1)$ and that $\{f_n'(z)\}$ converges to $f'(z)$ uniformly on every $B(0,r)$, $r < 1$.

II.3 Let $p_n(z) = \displaystyle\sum_{k=0}^{n} \frac{z^k}{k!}$.

(i) Show that for any $r > 0$ there exists an $N \in \mathbb{Z}$ such that for all $n > N$ $p_n(z)$ has no zeros in $B(0,r)$.

(ii) Show that for any $r > 0$ and any $n \geq 0$ there exists z with $|z| = r$ such that $|p_n(z)| = |e^z|$.