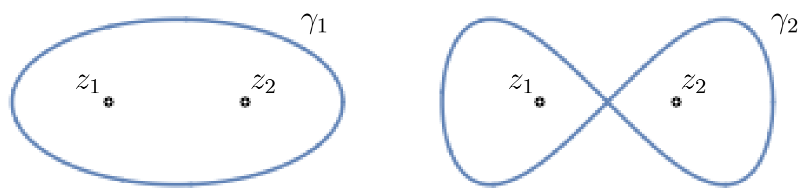
Ph.D. Comprehensive Examination
Complex Analysis

August 2016

Part I. Do three of these problems.

I.1 Let $G \subset \mathbb{C}$ be open, z_1, $z_2 \in G$ distinct points, and γ_1, γ_2 closed smooth paths in G as indicated in the figure, each homotopic in G to a constant and oriented so that its index



with respect to z_2 is 1. Suppose that f is meromorphic in G with poles only at z_1 and z_2 and such that

$$\int_{\gamma_1} f(z)dz = 2\pi i, \qquad \int_{\gamma_2} f(z)dz = -6\pi i.$$

Find the residues of f at z_1 and z_2.

I.2 Let f be defined on some open set $U \subset \mathbb{C}$, assume f^2 and f^3 are holomorphic on U. Show that f is holomorphic on U.

I.3 Show that the punctured unit disk $B(0,1) \setminus \{0\}$ in $\mathbb{C}$ is not conformally equivalent to the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

I.4 Consider $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^6 + 4z^2 e^{z+1} - 3$. Determine the number of zeroes (counted with multiplicities) of the function f in the unit disk $B(0,1) := \{z \in \mathbb{C} : \ |z| < 1\}$.

Part II on next page

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part II. Do two of these problems.

II.1 For an open set $\Omega \subset \mathbb{C}$ let $\mathcal{H}(\Omega, \Omega)$ be the set of all holomorphic functions from Ω into Ω. Suppose Ω is open, bounded, connected, with $0 \in \Omega$.

(1) Show that there are positive numbers M and r such that

$$\left|\frac{d^k h}{dz^k}(0)\right| \leq \frac{k!M}{r^k} \quad \text{for all} \quad h \in \mathcal{H}(\Omega, \Omega) \quad \text{and all} \quad k \in \mathbb{N} \cup \{0\}.$$

(2) Let $f \in \mathcal{H}(\Omega, \Omega)$ be such that

$$f(0) = 0, \quad f'(0) = 1.$$

Show that $f(z) = z$.

Hint: For Part (2), observe that $f(z) = z + z^m g(z)$ for some $m > 1$ and some holomorphic g and that $f \circ \cdots \circ f(z) = z + kz^m g(z) + \mathcal{O}(z^{2m-1})$ where $\mathcal{O}(z^{2m-1})$ represents a term vanishing to at least order $2m - 1$.

II.2 Let f and g be entire functions and suppose there exists $R > 0$ such that $|f(z)| \leq |g(z)|$ whenever $|z| > R$. Prove that there exist two polynomials p and q, not both zero, such that

$$p(z)f(z) + q(z)g(z) = 0, \qquad \forall\, z \in \mathbb{C}.$$

II.3 Does there exist a function $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ which is holomorphic and satisfies

$$|f(z)| \geq \frac{1}{\sqrt{|z|}}$$

for all $z \in \mathbb{C}\backslash\{0\}$? Either give an example or prove that no such function exists.