Ph.D. Comprehensive Examination in Complex Analysis
Department of Mathematics, Temple University

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Part I: Do three of the following problems

1. Let $u(x,y)$ be a polynomial of degree n that is harmonic in $\mathbb{C}$. Show that $u(x,y) = \Re(f(z))$ where $f(z)$ is a polynomial of degree n.

2. Let $f(z)$ be an analytic function from the open unit disc D onto D. Let $M(r) = \max\{|f(z)| : \ |z| = r\}$.

(i) Show that $M(r)$ is a strictly increasing function of r.

(ii) Show that $\lim_{r \to 1-} M(r) = 1$.

3. Use the calculus of residues to find the principal value of $\int_{-\infty}^{\infty} \frac{dx}{x(2x^2 - 2x + 1)}$. Here the principal value of $\int_{-\infty}^{\infty} f(x)dx$ means $\lim_{\epsilon \to 0+} \left(\int_{-\infty}^{-\epsilon} f(x)dx + \int_{\epsilon}^{\infty} f(x)dx \right)$.

4. (i) Find a bijective conformal mapping from $\mathbb{C} - [1, \infty)$ to the open unit disc D.

(ii) Find a conformal mapping from $\mathbb{C} - [0, 1]$ onto the open unit disc D. Can this map be bijective? Why or why not?

Part II: Do two of the following problems

1. Let $G \subset \mathbb{C}$ be a region in $\mathbb{C}$ and let $I = [a, b]$ be a line segment, $I \subset G$. Let $f(z)$ be continuous in G and analytic in $G - I$. Show that $f(z)$ is analytic in G.

2. Let $f(z)$ be analytic in $\{z : 0 < |z| < 1\}$ except for a sequence of isolated non removable singularities $\{z_n\}$ with $\lim_{n\to\infty} z_n = 0$. Show that any $w \in \mathbb{C}$ and any $\epsilon, \delta > 0$, there exists a $z \neq z_n$ with $0 < |z| < \delta$ such that $|f(z) - w| < \epsilon$.

3. Let G be a region in $\mathbb{C}$ that contains the closed unit disc $\bar{D}$ and let $f(z, t)$ be a continuous function on $G \times [0, 1]$ that is analytic in z.

(i) Show that $f'(z, t)$ is continuous on $G \times [0, 1]$, where $f'(z, t)$ denotes $\dfrac{\partial}{\partial z} f(z, t)$.

(ii) Suppose $f(z, t) \neq 0$ for z with $|z| = 1$ and any $t \in [0, 1]$. Show that $f(z, 1)$ has the same number of zeroes in D, counting multiplicities, as $f(z, 0)$.