Ph.D. Comprehensive Examination in Complex Analysis
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Part I: Do three of the following problems

1. Let f(z) be an entire function. Suppose there exists $z_0 \in \mathbb{C}$ and $R > 0$ such that $B(z_0; R) \bigcap f(\mathbb{C}) = \emptyset$, where $B(z_0; R) = \{z \in \mathbb{C} : |z - z_0| < R\}$. Show that $f(z)$ is a constant function.

2. Let U be the open unit disc and let $f(z)$ be a nonconstant function analytic on some open set containing $\bar{U}$. Suppose $f(\partial U) \subset \partial U$. Prove that

(a) $f(U) \subset U$;

(b) $f(z)$ has a zero in U.

3. Use the residue theorem to evaluate $\displaystyle\int_0^\infty \frac{\cos x - 1}{x^2(x^2+1)}\, dx$.

4. Suppose $f(z)$ has a pole at $z = a$.

(a) Prove that for any $\delta > 0$ there exists an $R > 0$ such that

$$\{z \in \mathbb{C} : |z| > R\} \subset f(\text{ann}(a; 0, \delta)),$$

where $\text{ann}(a; 0, \delta)) = \{z \in \mathbb{C} : 0 < |z - a| < \delta\}$.

(b) Prove that $e^{f(z)}$ has an essential singularity at $z = a$.

Part II: Do two of the following problems

1. Let G be a simply connected domain and let $f(z)$ be analytic in G.

(a) Prove that there exists a function $F(z)$ analytic in G such that $F'(z) = f(z)$.

(b) Suppose further that $f(z) \neq 0$ for any $z \in G$. Prove that there exists a function $g(z)$ analytic in G such that $f(z) = e^{g(z)}$ and a function $h(z)$ analytic in G such that $f(z) = (h(z))^3$.

2. (a) Prove that $\cos \pi z = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right)$.

(b) Prove that $\pi \tan \pi z = -\sum_{n=0}^{\infty} \frac{8z}{4z^2 - (2n+1)^2}$ for every $z \neq n + \frac{1}{2}$, $n \in \mathbb{Z}$

3. Let $\{f_n(z)\}$ be a sequence of analytic functions in G. Suppose $\{f_n(z)\}$ converges to a function $f(z)$ uniformly on every compact subset of G and that $f(z)$ has no zeros in G.

(a) Let $B(a; R)$ be an open disc such that $\bar{B}(a; R) \subset G$. Prove that there exists an $N > 0$ such for any $n > N$ $f_n(z)$ has no zeros in $B(a; R)$.

(b) Let K be a compact subset of G. Prove that there exists an $N > 0$ such for any $n > N$ $f_n(z)$ has no zeros in K.

(c) Let $S = \{z \in G : \ f_n(z) = 0 \text{ for some } n \in \mathbb{N}\}$. Prove that S has no accumulation points in G.