August, 2005

Comprehensive Examination

Department of Mathematics

Complex Analysis

Part I: Do three of the following problems

1. Let $f(z) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are real-valued, be an entire function. Suppose $u(x, y) = x^3 - 3x + \alpha xy^2$ for some real number α. Determine all possible values of α and find a function $f(z)$ that corresponds to each α (note that $f(z)$ is uniquely determined up to an additive constant).

2. Let $f(z)$ be analytic in an open set G that contains the closed unit disc $\bar{D}$. Suppose $|f(z)| = 1$ for any z with $|z| = 1$.
(a) Show that $f(z)$ maps $\bar{D}$ to $\bar{D}$.
(b) Suppose $f(z)$ has no zeros in D. Show that $f(z) = e^{i\theta}$, a constant function of absolute value 1.
(c) More generally, show that $f(z)$ has only finitely many zeros in D and if z_1,..., z_n are the zeros of $f(z)$ in D, listed with multiplicities, then

$$f(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z_k} z}.$$

3. Show that $\int_{-\infty}^{\infty} \dfrac{e^{wx}}{x^2 - x + 1}\, dx$ converges for any $w \in \boldsymbol{C}$ with $Re(w) \le 0$ and use the calculus of residues to determine

$$\int_{-\infty}^{\infty} \frac{e^{-ax} \cos bx}{x^2 - x + 1}\, dx \text{ and } \int_{-\infty}^{\infty} \frac{e^{-ax} \sin bx}{x^2 - x + 1}\, dx \text{ where } a, b \in \boldsymbol{R},\ a \ge 0.$$

4. (a) Give an example of a conformal map from the unit disc D onto $\boldsymbol{C} \setminus \{0\}$. Hint: first transform D into the upper half-plane.
(b) Show that there exists no conformal map from $\boldsymbol{C} \setminus \{0\}$ onto D.

Part II: Do two of the following problems

1. Let G be a simply connected domain, $G \neq \mathbb{C}$, and let $a \in G$. Let $f(z)$ be a holomorphic function from G to G such that $f(a) = a$.

(a) Show that $|f'(a)| \leq 1$.

(b) Show that $|f'(a)| = 1$ if and only if $f(z)$ is bijective.

2. Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$ such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$. Let $\Pi = \{z \in \mathbb{C} : \ 0 \leq Re(z) \leq 1, \ 0 \leq Im(z) \leq 1\}$. Suppose further that $f(z)$ has no zeros or poles on $\partial\Pi$. Let N and P denote respectively the numbers of zeros and poles of $f(z)$ in Π.

(a) Show that $P = N \geq 2$.

(b) Let z_1,..., z_N and w_1,..., w_N be the zeros and the poles of $f(z)$ in Π, listed with multiplicities. Show that

$$z_1 + ... + z_N - w_1 - ... - w_N = m + ni \text{ for some } m, n \in \mathbb{Z}.$$

3. (a) Construct an entire function with a zero of order n^3 at every positive integer n and no other zeros. Justify every statement you make.

(b) Construct a meromorphic function with a simple pole with residue 1 at $\sqrt{n}$ for every positive integer n and no other poles. Justify every statement you make.