Ph.D. Comprehensive Examination
Complex Analysis Section
August 2003

Part I. Do three (3) of these problems.

I.1. Let $f : [0, 1] \to \mathbb{C}$ be continuous. Define $g : \mathbb{C} \setminus [0, 1] \to \mathbb{C}$ by

$$g(z) = \int_0^1 \frac{f(t)}{z - t} dt.$$

Show that g is holomorphic.

I.2. Let $n \geq 2$ be an integer. Evaluate

$$\int_0^\infty \frac{dx}{1 + x^n}$$

Hint: $e^{i\pi/n}$ lies between the positive real axis and the ray $t \mapsto e^{2\pi i/n}t$, $t > 0$.

I.3. How many solutions are there to $\sin z = z$? (Verify)

I.4. Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and bounded. Prove that u is constant.

Part II. Do two (2) of these problems.

II.1. Let Ω be open connected in $\mathbb{C}$, $f, g : \Omega \to \mathbb{C}$ holomorphic with g not constant. Suppose there is $h : g(\Omega) \to \mathbb{C}$ continuous such that $f = h \circ g$. Prove that h is holomorphic. Hint: The property of being holomorphic is local.

II.2. Let Ω and R be open, bounded, and connected, each with boundary consisting of finitely many circles. Let $f : \overline{\Omega} \to \mathbb{C}$ be continuous, holomorphic in Ω, and nonconstant. Suppose that $f(\partial\Omega) \subset \partial R$ and that for every bounded component Q of $\mathbb{C} \setminus R$ there is $q \in Q$ not in $f(\overline{\Omega})$. Prove that $f(\Omega) \subset R$. Hint: use the Maximum Principle.

II.3. Let $\mathcal{F} = \{f : D \to D \mid f \text{ is holomorphic}\}$, where $D = \{z \mid |z| < 1\}$. Let $L = \sup_{f \in \mathcal{F}} |f''(0)|$. Show there is $f \in \mathcal{F}$ such that $f''(0) = L$.