PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

August 1996

Part I. Do three (3) of these problems.

I.1. Let $f(z) = u(x, y) + iv(x, y)$ be an entire function. Suppose $u(x, y)$ is a function of x alone. Show that $f(z) = az + b$ where a and b are constants and $a \in \mathbb{R}$.

I.2. Let $D = \{z : \ |z| < 1\}$ be the open unit disc. Find the image of D under the map $f(z) = e^{\frac{i-iz}{z+1}}$.

I.3. Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of functions analytic on an open unit disc D, and let $f_n(z) = \sum_{k=0}^{\infty} a_{nk} z^k$ be the Taylor series expansion of $f_n(z)$ on D. Suppose the sequence $\{f_n(z)\}_{n=1}^{\infty}$ converges to a function $f(z)$ on $\bar{D}$ and that convergence is uniform on every compact subset of D. Show that for every $k \geq 0$ $a_k = \lim_{n\to\infty} a_{nk}$ exists and that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on D.

I.4. Evaluate $\int_0^\infty \frac{log^2 x}{x^2+1} dx$ and $\int_0^\infty \frac{log^4 x}{x^2+1} dx$.

Part II. Do two (2) of these problems.

II.1. i) Let $f(z)$ be an entire function. We say that $f(z)$ has a removable singularity, a pole or an essential singularity at ∞ if $f(\frac{1}{z})$ has respectively a removable singularity, a pole or an essential singularity at 0. Show that $f(z)$ has a removable singularity or a pole at ∞ if and only if it is a polynomial.

ii) Let $f(z)$ be an entire function that is finite-to-one (i. e. for any $w \in \mathbb{C}$ the number of solutions $f(z) = w$ is finite). Show that $f(z)$ is a polynomial.

II.2. Let G be a simply connected region other than $\mathbb{C}$, and let $a \in G$. Let $f : G \to G$ be an analytic function such that $f(a) = a$. Show that $|f'(a)| \leq 1$. Moreover, if $|f'(a)| = 1$, show that $f(z)$ is one-to-one and onto.

II.3. i) Give a definition of a simply connected region.

ii) Let G be a simply connected region, and let $f : G \to \mathbb{C}$ be a one-to-one analytic map. Show that $f(G)$ is also simply connected.