PH.D. COMPREHENSIVE EXAMINATION
ALGEBRA SECTION

January 1995

Part I. Do three (3) of these problems.

I.1. If a subgroup G of the symmetric group S_n contains an odd permutation, then $|G|$ is even and exactly half the elements of G are odd permutations.

I.2. Let R be a commutative ring with no nonzero nilpotent elements (that is, $a^n = 0$ implies $a = 0$). If the polynomial $f(X) = a_0 + a_1X + \ldots + a_mX^m$ in $R[X]$ is a zero-divisor (that is, $g(X)f(X) = 0$ for some nonzero polynomial $g(X) \in R[X]$), prove that there is an element $b \neq 0$ in R such that $ba_0 = ba_1 = \ldots ba_m = 0$.

I.3. Let V be a finite-dimensional vector space over a field F. An endomorphism ϕ of V is called a *pseudoreflection* if $\phi - 1$ has rank at most 1. Prove:

a) ϕ is a pseudoreflection precisely if there exists a basis of V such that the matrix of ϕ has the form

$$\begin{bmatrix} * & * & * & \ldots & * \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

b) Show that the Jordan canonical form of a pseudoreflection ϕ is

$$\begin{bmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & 1 \end{bmatrix}.$$

I.4. Let $F \supseteq K$ be an algebraic extension of fields and let R be a subring of F with $R \supseteq K$. Show that R is a field.

Part II. Do two (2) of these problems.

II.1. Let G be a finite group and let H be a proper subgroup of G. Show that G is not the set-theoretic union of all conjugates of H.

II.2. Let K be the splitting field over the rationals $\mathbb{Q}$ for the polynomial $f(x)$. For each of the following examples, find the degree $[K : \mathbb{Q}]$, determine the structure of the Galois group $G(K/\mathbb{Q})$, describe its action on the roots of $f(x)$ and identify the group.

a) $f(x) = x^4 - 3$
b) $f(x) = x^4 + x^2 - 6$

II.3. Let G be a group of order $165 = 3 \cdot 5 \cdot 11$. Prove:

a) G has a normal Sylow 11-subgroup, say C.
b) G/C is cyclic. (HINT: Show that every group of order 15 is cyclic.)
c) G has normal subgroups of orders 33 and 55.