August, 1998

Comprehensive Examination **Department of Mathematics**

ALGEBRA

PART I: Do three of the following problems.

1. Let G be a group.
   (a) Show that G is finite iff G has only finitely many distinct subgroups.
   (b) Show that G has exactly 3 distinct subgroups iff G is cyclic of order p^2 for some prime p.
2. Let A be a real $r \times r$-matrix satisfying $A^n = I$ for some $n > 0$. Prove: $\det A = (-1)^m$, where m is the multiplicity of -1 as root of the characteristic polynomial of A.
3. Let R be a ring and let N be an ideal of R such that every element of $x \in N$ is nilpotent, that is, $x^t = 0$ for some t. Show that, under the canonical map $R \to R/N$, the group of units $U(R)$ of R maps *onto* the group of units $U(R/N)$ of R/N. (Recall that a *unit* of a ring R is an invertible element of R.)
4. Let $F \supseteq K$ be an algebraic extension of fields and let R be a subring of F with $R \supseteq K$. Show that R is a field.

PART II: Do two of the following problems.

1. Let G be a group of order pqr with distinct primes p, q, and r. Show that G is not simple.

2. Let $R = K[x, y]$ be the ring of polynomials in two variables x and y with coefficients in the field K, and let $f(x, y) \in R$.

   (a) Show that the principal ideal of R that is generated by $f(x, y)$ is prime if and only if the polynomial $f(x, y)$ is irreducible.

   (b) Show that the ideal of R that is generated by x and $f(x, y)$ is maximal if and only if the polynomial $f(0, y)$ is irreducible in $K[y]$.

3. Let F be the splitting field of $x^6 - 3$ over $\mathbf{Q}$.

   (a) Show that $[F : \mathbf{Q}] = 12$.

   (b) Let $G = Gal(F/\mathbf{Q})$. Show that there exist a normal subgroup H of G of order 6 and a subgroup K of G of order 2 such that G is a semidirect product of H and K.

   (c) Determine whether the subgroup H of Part (b) is abelian.