PH.D. COMPREHENSIVE EXAMINATION
ABSTRACT ALGEBRA SECTION

August 1996

Part I. Do three (3) of these problems.

I.1. Let $(G, \cdot)$ be a group with binary operation $\cdot$, and let a be a fixed element of G. Define a binary oparation $*$ on G by setting $x * y = x \cdot a \cdot y$, for x, $y \in G$. Show that $(G, *)$ is a group and that it is isomorphic to $(G, \cdot)$.

I.2. Let $R = \mathbb{Z}[x]$ be the ring of polynomials with coefficients in $\mathbb{Z}$, the ring of integers; and let $p \in \mathbb{Z}$ be a prime number. Show that pR, the ideal generated by p, is a prime ideal of R. Is pR maximal? If so, explain why. If not, find the generators of a maximal ideal which contains pR.

I.3. a) Prove that a finite field must be of order p^n for some prime p and some positive integer n

b) Show that for any such p and any such n there exists a field of order p^n.

I.4. Let P_n denote the set of real polynomials of degree $\leq n$, and let $T : P_n \to P_n$ be defined by $T(p(t)) = (t-1)p'(t) + p(1)$.

a) Prove that T is linear.
b) Find the matrix of T with respect to the basis $\{1, t, t^2, \ldots, t^n\}$ for P_n.
c) Find a basis for P_5 with respect to which the matrix of T is diagonal.

Part II. Do two (2) of these problems.

II.1. Let G be a group and let H be a subgroup of some finite index n in G.

a) Show that H contains a normal subgroup of G whose index divides $n!$. (Hint: Consider the action of G on G/H by right multiplication.)
b) Show that $K = \bigcap_{a \in G} aHa^{-1}$ is a normal subgroup of G and that any other normal subgroup of G which is contained in H is contained in K.
c) Show that if K, as defined above, consists of only of the identity element, e, then G can be embedded in a permutation group of order $n!$.

II.2. Let V be a vector space of dimension n over a field K. We call a linear transformation $T : V \to V$ nilpotent if there exists an integer N such that $T^N = 0$, the zero map. Let N be the smallest such integer.

a) Show that if T is nilpotent, then $T^k(V) \subset T^{k-1}(V)$ and $\dim T^k(V) < \dim T^{k-1}(V)$ for every integer k, $1 \leq k \leq N$.

b) Show that if T is nilpotent, then $T^n = 0$ (i.e., $N \leq n$).

c) Prove that if T is nilpotent, then $(T - I)$ is invertible, where I is the identity map.

II.3. Let p be a prime number, and let K be a field of characteristic $q \neq p$ which contains all p^{th} roots of unity. Let a be an element of K which is not a p^{th} power in K, and let α be a p^{th} root of a in an algebraic closure $\overline{K}$ of K.

a) Determine $[K(\alpha) : K]$.

b) Determine the Galois group $\mathrm{Gal}(K(\alpha); K)$ of $K(\alpha)$ over K.

c) Determine all extensions of the field K contained in the field $K(\alpha)$.